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LETTER TO THE EDITOR

Spin-glass field theory in the condensed phase continued to below $d = 6$

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Abstract. Field theory has been held back from being able to describe the spin-glass condensed phase in physical dimensions by the strong infrared divergences of its (bare) propagators with small overlaps, e.g. the zero overlap replicon propagator $G_{\mathrm{R}}^{00}(p) \sim p^{-4}$. Here we examine the effect of fluctuations at the one-loop level on the equation of state for the Parisi order parameter $q(x)$. We find that above $d = 6$, the one-loop term does not change the analytical behaviour of $q(x) \sim x$. Below $d = 6$, the loop contribution becomes dominant and radically changes the classical behaviour into $q(x) \sim x^{\rho}$, $\rho = 3/(d-3) \sim 1 + \varepsilon/3$. Besides, since the infrared divergences are driven by the small-x behaviour, we also get $G_{\mathrm{R}}^{00}(p) \sim p^{-4+2\varepsilon/3}$. This remark is also valid away from T_{c} since the weakening of infrared singularities by fluctuations occurs solely via the (stronger) vanishing of $q(x)$ near zero overlaps. Consistently taking account of fluctuations should thus allow a fully detailed field theory description of the condensed phase below $d = 6$.

The possibility of setting up a meaningful field theory of the condensed phase of spin glasses in physical dimensions has tantalised physicists for more than ten years. This problem could be readily dismissed so long as a spin glass could be assumed to exist only in high dimensions, for Parisi [1] had taught us how to build a mean-field solution that, with all its unusual features (breaking of ergodicity or of replica symmetry), was shown to be stable [2]. As for experiments, they could always be considered as belonging to the realm of non-equilibrium dynamics. The situation became dramatic when most doubts were lifted in 1985 by the Ogielski [3] Monte Carlo calculation [4, 5]: the Ising spin glass did undergo a phase transition in three dimensions.

A nearest-neighbour Ising spin with Gaussian random bonds does not allow any other starting point for field theory than the Parisi mean-field one, around which a loop expansion is to be organised. And changing from a Gaussian to a discrete (say $\pm J$) distribution does not leave enough room† to allow for a replica symmetric starting point. So the conclusion is inescapable: one has to work with Parisi's ansatz and fluctuations around it. This conclusion has recently been reinforced by the findings of two groups [6, 7] that suggest the existence of ergodicity breaking in $d = 3$, via numerical simulations.

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† In the best of the cases (and excluding singular distributions) the domain of dimensionality where a replica symmetric solution (in the tree approximation) is stable, remains confined to $d < \frac{8}{3}$! This is for a $\pm J$ distribution, truncated to its first two cumulants [8].

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With that starting point established, one has then to face all the contradictions that appear as direct consequences of the standard field theory.

Consider the 'bare' propagators first, and the simplest of all

$$G^{\alpha\beta;\alpha\beta}(p) \equiv \langle \delta\varphi^{\alpha\beta}(\boldsymbol{p})\delta\varphi^{\alpha\beta}(-\boldsymbol{p})\rangle_0. \tag{1}$$

Here the spin-glass field is $\varphi^{\alpha\beta}$ (of wavevector $\boldsymbol{p}$), α and β are a pair of replicas (with overlap $\alpha \cap \beta = x$), and $\delta\varphi_j^{\alpha\beta}$ is the shift away from its mean-field average value $\langle\varphi^{\alpha\beta}\rangle_{\mathrm{MF}} \equiv q(x)$. In (1) the average is computed with the Lagrangian reduced to its quadratic part in $\delta\varphi^{\alpha\beta}$. In our notation this is $G_{1,1}^{x,x}(p)$. It has been shown [9] to be related to the two-states (l, l') correlation function

$$C_{ll'}(p) = N^{-1}\sum_{ij}\exp[\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)]\langle\sigma_i\sigma_j\rangle_l\langle\sigma_i\sigma_j\rangle_{l'} - N\delta_{p;0}q_{ll'} \tag{2}$$

via

$$G_{1,1}^{x,x}(p) = C_{ll'}(p) \tag{3}$$

together with

$$q(x) = q_{ll'} \tag{4}$$

which identifies the Parisi order parameter and the (bond-averaged) overlap function [10]. This propagator $G_{1,1}^{x,x}(p)$, and more complicated ones that involve two or three replica overlaps, have an infrared behaviour [11] in $1/p^2$ when the overlaps are maximal, i.e. when those propagators pertain to a single (pure) state [12]. For example, one finds [9, 11]

$$G_{1,1}^{x_1,x_1}(p) = \frac{1}{p^2}\left[1 + \frac{x_1}{p^2 + x_1} + \left(\frac{x_1}{p^2 + x_1}\right)^2\right] \tag{5}$$

for $p^2 \ll x_1^2$ (and also for $p^2 \gg x_1^2$), with $q(x_1) = q_{ll}$ the maximal value of the Parisi order parameter (and x_1 the breakpoint value beyond which $q(x)$ remains constant). Such an infrared behaviour would leave Parisi order intact in $d = 3$. Unfortunately, for small overlaps, the infrared singularity grows stronger, and for $x = 0$ one gets [11, 13]

$$G_{1,1}^{0,0}(p) = \frac{\pi}{4p^4} \tag{6}$$

a fluctuation bound to destroy Parisi order (in $d \leqslant 4$). This is only one example out of many in a complicated set of propagators [11, 14].

Another feature that has emerged recently [9], as a consequence of a Schwarz inequality over state correlations, says, in substance, that

$$C_{ll}(r) > C_{ll'}(r). \tag{7}$$

To view it in its most dramatic way let us take $q_{ll'} = 0$. We then obtain

$$q^2(x_1) + \frac{\text{constant}}{r^{d-2}} > \frac{\text{constant}}{r^{d-4}} \tag{8}$$

i.e. for a distance $r \sim 1/x_1 \gg 1$, we get, with $q(x) = x/2$,

$$x_1^2/4 + \text{constant}\ x_1^{d-2} > \text{constant}\ x_1^{d-4} \tag{9}$$

an inequality violated for $d < 6$ as $x_1 \ll 1$.

Obviously this is telling us that below $d = 6$, fluctuations are taking over. And indeed, this is to be expected if we assume that one is working with a system that is critical at all $T < T_c$ [4]. So it becomes essential, when solving the equation of state for the spin glass, to take into account loops below a critical dimension above which the Parisi solution is exact in substance.

Let us thus start from the Lagrangian kept to its $w\varphi^3 + u\varphi^4$ terms [15, 16] in the so-called Parisi [1] approximation and let us work out the equation of state up to one loop. We find

$$0 = \tau q(x) - \frac{w}{2}\left(xq^2(x) + \int_0^x \mathrm{d}t\, q^2(t) + 2q(x) \int_x^1 \mathrm{d}t\, q(t) \right) + \frac{2}{3}uq^3(x)$$
$$- \frac{w}{2} L_w(x) + uq(x)L_u(x) + \ldots \tag{10}$$

with†

$$L_w(x) = xG_{x1}^{xx} + \int_0^x \mathrm{d}t\, G_{x1}^{tt} + 2\int_x^1 \mathrm{d}t\, G_{x1}^{xt} \qquad L_u(x) = G_{11}^{xx} \tag{11}$$

and, for shorthand, $G = \Sigma_p G(p)$. Let us now examine the mean-field and the fluctuation regimes as dimensionality varies.

(i) *Mean-field regime (classical).* Let us first forget the loop terms. How is the mean-field answer obtained from the equation of state (10)? The way it is derived is by repeated x derivatives (and divisions by $\dot{q}(x) \neq 0$) that in the end leaves the equation

$$0 = -wx + 2uq(x). \tag{12}$$

The w terms in the equation of state are polynomials of third degree in x and in (12) it is its *highest-degree* term (typically the $wxq^2(x)$ of (10)) that is matched with the u term. Terms *of lesser degree* (but dominant as $x \ll x_1$) are taken care of by the constants of integrations that take back from (12) to (10). Likewise if we now evaluate the loops, we obtain for the u-loop term, keeping to its most dangerous contribution that comes from the 'replicon' [15, 16] sector‡,

$$uq(x)L_u^{\mathrm{Rep}}(x) \sim uq(x)\left(\frac{q^4(x)}{x^2}\right)\Lambda^{d-6} \qquad d > 6 \tag{13}$$

where the explicit form [11, 14, 18] of $(G_{11}^{xx})_{\mathrm{Rep}}$ has been used. Here Λ plays the role of an ultraviolet cutoff. Hence classically, as $q(x) \sim x$, the u-loop term gives the same analytic behaviour as the $uq^3(x)$ term (the 'replicon' component of the w-loop also has the same behaviour). This is expected, for loops in the classical regime only change values of the parameters, not the analytical behaviour. As above, terms of lesser degree from the u-loop are left aside to satisfy integration boundary conditions.

(ii) *Fluctuation-dominated regime (non-classical).* The above regime survives as long as $d \geq 6$. Below $d = 6$, letting $\varepsilon = 6 - d$, one finds for the same u-loop

$$uq(x)L_u^{\mathrm{Rep}}(x) \sim uq(x)\left(\frac{q(x)^{d-2}}{\varepsilon x^2}\right). \tag{14}$$

† For $G^{\alpha\beta;\gamma\delta}$ one writes [14] $G_{t_1;t_2}^{x;y}$ with $\alpha \cap \beta = x$, $\gamma \cap \delta = y$, $t_1 = \max(\alpha \cap \gamma; \alpha \cap \delta)$, $t_2 = \max(\beta \cap \gamma; \beta \cap \delta)$, ultrametricity [2, 17] leaving only three distinct indices.

‡ In zero (infinitesimal) magnetic field. In non-zero field the replicon propagator becomes well behaved ($1/p^2$ in the classical regime) and the most dangerous one is the 'longitudinal-anomalous' propagator [11, 14] which is far more complicated to handle analytically.

Here we see clearly that now the u-loop dominates at small x over the $uq^3(x)$ term (for $x \sim 0$ if $q(x) \sim x$ then $q(x)L_u^{\mathrm{Rep}}(x) \sim q(x)^{3-\varepsilon}$). As above, we leave aside terms of lesser degree from the u-loop, and match (14) with typically $wxq^2(x)$. We obtain (the w-loop offers qualitatively the same behaviour)

$$q(x) \sim \varepsilon^{1/(d-3)} x^{\rho} \qquad \rho = \frac{3}{d-3} \sim 1 + \frac{\varepsilon}{3} \tag{15}$$

which exhibits *non-classical* behaviour when $d < 6$. Note that the slope of $q(x)$ vanishes at $x = 0$, a result that we learn has just been obtained, via a $1/d$ expansion, at $d = 6$, by Georges, Mézard and Yedidia [19]. With (15) one finds that the actual u-loop behaviour is $q(x)L_u^{\mathrm{Rep}}(x) \sim q(x)^{3-\varepsilon/3}$.

Let us now return to the replicon component of $G_{1,1}^{x,x}(p)$. Its infrared divergence is governed by the small-x behaviour of $q(x)$. When $q(x) \sim x^{\rho}$ as in (15), one derives then, for $p \ll x_1$,

$$G_{11}^{00}(p) \sim \frac{1}{p^{2+2/\rho}} = \frac{1}{p^{4-2\varepsilon/3}}. \tag{16}$$

Hence the infrared divergence is weakened. Turning to *Schwarz inequality* we see that the right-hand side of (8) decreases now more rapidly (as $r^{2/\rho+2-d}$). The left-hand side behaviour is determined by solving the equation of state around the breakpoint x_1. In that region all propagators are in $1/p^2$ and the one-loop terms are unlikely to drastically modify the classical behaviour $q(x_1) \sim x_1$. Assuming that, and further that $x_1 \ll 1$, one can then check that (16) improves the inequality situation (9) and that the self-consistent solution evoked below satisfies it.

The above results thus open the way to build propagators dressed by fluctuations that no longer destroy Parisi order in physical dimensions.

A few remarks are finally in order.

(i) The above results bear no relationship to any proximity of T_c. They have to do with the fact that here, contrary to standard critical phenomena where the order parameter vanishes at T_c only, *there is always a region of very small order parameter*, which drives the infrared singularities. In that sense the system is *critical at all temperatures* (*below* T_c).

(ii) The above calculation is not self-consistent since the replicon u-loop has been evaluated with the mean-field values of the replicon masses [2, 13]

$$\lambda_{\mathrm{Rep}}(k_1, k_2, x) = M(k_1) + M(k_2) - 2M(x) \qquad M(x) = uq^2(x). \tag{17}$$

One may try to be more sophisticated, and *self-consistently* take the u-loop into account by letting $M(x) \sim q(x)^L$. This leads, under the procedure described above, to $L = 2 - \varepsilon$, $\rho = 1 + \varepsilon$, $G_{11}^{00}(p) \sim p^{-4+\varepsilon}$ as compared with the previous set $L = 2 - \varepsilon/3$, $\rho = 1 + \varepsilon/3$, $G_{11}^{00}(p) \sim p^{-4+2\varepsilon/3}$.

(iii) One should not perhaps put too much weight on these figures, for there are many points that have still to be clarified. For example, back to (15), one sees that $q(x)$ contains a non-integer power of ε. Besides, we have seen that, at the one-loop level, the u-loop contributes although from naïve power counting the u-coupling is irrelevant (but the u-coupling is also 'dangerous' [20] since it forces in replica symmetry breaking). Admittedly this anticipates that a full renormalisation group description of the spin-glass condensed phase may still remain at a distance and that much work will have to be accomplished to put our understanding at the level of standard field theories of critical phenomena.

Nevertheless we believe that we have uncovered a way in which one can make sense at $d<6$ (and hopefully at the physical dimension) of the field-theoretical description of the spin-glass condensed phase, justifying the obstinate ground work put into understanding the 'bare' spin-glass properties in the last few years.

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